Proceedings of the 27th Conference
on Decision and Control
Austin, Texas • December 1988

Remarks on H^∞ Optimization of Multivariate Distributed Systems

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Abstract

This is a summary of our work [6], where we study the H^∞ optimization problem for multivariable distributed systems. This note is based on several previous papers and basically uses the skew Toeplitz framework developed in [1], [4a,b].

1 Introduction

In recent years there has been large literature concerned with the H^∞ optimization of linear systems, to which we refer the reader to the book [9] for a complete set of references. In this paper we study the problem of the H^∞-optimization for multivariable distibuted systems.

Motivations leading to the H^∞ optimization in systems theory lie in the most natural problems of control engineering such as robust stabilization, sensitivity minimization, and model matching. It can be shown that, in the sense of H^∞ optimality, these problems are equivalent, and can be stated as one *standard problem*, to which we once again refer the reader to [9] for further discussions and details. In this context the standard problem can be reduced to finding the singular values of a certain operator (the so-called **four block operator**) which will be defined below. Depending on the specific problem considered, the corresponding four block operator can be simplified to a 2-block or a 1-block operator.

This note is based on several previous papers [1], [4a,b], [5], [7], [8], [11], and basically employs the skew Toeplitz framework of [1] to study the standard problem. This is only a sketch of our full work and we refer the reader, for complete details, to [6].

The paper is organized as follows. In the next section we set up some notation and give some background on the ideas taken from previous work. Then we specialize to multivariate two block case and give its motivations. In section four we obtain our main result; and finally we summarize the results and mention future extensions of this work.

2 Preliminary Remarks

We now define the four block operator and make some preliminary remarks. In the paper all Hardy spaces are defined on the unit disc D in the standard way. We denote the unilateral shift by $S : H^2(\mathbf{C}^N) \longrightarrow H^2(\mathbf{C}^N)$. Let $W, F, G, H, M \in H^\infty(L(\mathbf{C}^N))$ be $N \times N$ matrices, where W, F, G, H have rational entries, and M is nonconstant inner. These matices are associated with the weighting matrices and the plant in the usual way of transforming the standard problem to the 4-block framework; see [9]. It is important to note that for rational weights and distributed stable plants this reduces to the kind of problem described below.

To M we associate the spaces $H(M) := H^2(\mathbf{C}^N) \ominus MH^2(\mathbf{C}^N)$ and $L(M) := L^2(\mathbf{C}^N) \ominus MH^2(\mathbf{C}^N)$. Define orthogonal projections $P : H^2(\mathbf{C}^N) \to H(M)$, $P_{L(M)} : L^2(\mathbf{C}^N) \to L(M)$, $P_{H^2} : L^2(\mathbf{C}^N) \to H^2(\mathbf{C}^N)$, and $P_{L^2 \ominus H^2} : L^2(\mathbf{C}^N) \to L^2(\mathbf{C}^N) \ominus H^2(\mathbf{C}^N)$.

In the study of standard H^∞ problem an operator unitarily equivalent to the following four block operator arises (see [3] and [4a,b]):

$$A := \begin{bmatrix} PW(S) & P_{L(M)}F(U) \\ G(S) & H(U) \end{bmatrix}.$$

Note that $A : H^2(\mathbf{C}^N) \oplus L^2(\mathbf{C}^N) \longrightarrow L(M) \oplus L^2(\mathbf{C}^N)$, where $U : L^2(\mathbf{C}^N) \to L^2(\mathbf{C}^N)$ is the bilateral shift.

The problem considered in [4a,b] and [6] is the computation of the singular values of the operator A. In this paper we will specialize to "two block" version where F and H are taken to be zero. It is important to note that the key step in the 1-block, 2-block, and 4-block problems is the computation of the Fredholm conditions for the invertibility of a certain skew Toeplitz operator which is essentially invertible. For the one block problem this is sufficient. In the two block we must also invert an ordinary Toeplitz operator, and for the four block two ordinary Toeplitz operators, as well as a skew Toeplitz operator. See [4b] for all the details.

3 The 2-Block Problem

As mentioned above, the 2-block problem is a special case of the more general four block problem with $F = H = 0$. But it has a special motivation, namely mixed sensitivity optimization. This corresponds to the control problem, where the standard weighted sensitivity minimization is modified by including the energy of the control action in the cost function as well as the energy of the error signal. For rigorous definition of mixed sensitivity optimization and more details we refer the reader to [2] and the references therein. This important paper has an operator theoratic approach to the problem. We should also note that in the nice paper [10] the SISO 2-block problem for general distributed systems is solved. Here we consider the multivariate distributed case in which one needs the full power of the matrix skew Toeplitz framework developed in [1].

In the computation of the singular values of the two block operator one ends up with a Hankel*Hankel + Toeplitz*Toeplitz type operator: A^*A (see [2] and also [10]). In order to illustrate that our set-up agrees with the usual definition of Hankel and Toeplitz operators, and also introduce some facts which will be used in the next section, we prove the following well-known proposition for the convenience of the reader. Note that $W(S)$ can be seen as the multiplication operator by $W(\zeta)$ for $\zeta \in \partial D$ the unit circle, and similarly for $G(S)$. With a slight abuse of the notation we write W for both $W(S)$ and $W(\zeta)$, and similarly for G.

Proposition 1. Let $H_{M^*W} : H^2(\mathbf{C}^N) \to L^2(\mathbf{C}^N) \ominus H^2(\mathbf{C}^N)$ be the Hankel operator defined by $H_{M^*W} = P_{L^2 \ominus H^2} M^* W$. Then $H^*_{M^*W} H_{M^*W} = W^* P W$.

Proof. Any $f \in H^2(\mathbf{C}^N)$ can be written as $f = f_1 + f_2$ where $f_1 \in H(M)$ and $f_2 \in MH^2(\mathbf{C}^N)$: $f_1 = Pf$ and $f_2 = MP_{H^2}f$.

[1]This work was supported by the National Science Foundation under Grant No. ECS-8704047 and by the Air Force Office of Scientific Research under Grant No. AFOSR-88-0020, and the Reaearch Fund of Indiana University

Therefore
$$\begin{aligned} M^*PWf &= M^*(Wf - MP_{H^2}M^*Wf) \\ &= M^*Wf - P_{H^2}M^*Wf \\ &= P_{L^2\ominus H^2}M^*Wf \end{aligned}$$

Hence $H_{M^*W} = M^*PW$ and $H^*_{M^*W}H_{M^*W} = W^*PW$. □

From this proposition, following Jonckheere and Verma [10], we deduce that $A^*A = W(S)^*PW(S) + G(S)^*G(S)$ is in the form Hankel*Hankel + Toeplitz*Toeplitz. Spectral properties of this type of operators are studied in [2] for multivariate finite dimensional systems, and in [10] for SISO distributed plants. We know that the H^∞ optimal performance is the supremum of the spectrum of A^*A. Also it is known that the spectrum of A^*A consits of the essential spectrum along with the isolated eigenvalues with finite multiplicities. A formula for the essential spectrum is given in [2] for multivariate finite dimensional systems, and in [10] for SISO infinite dimensional systems. Both of these pertain the 2-block case. In [4b] there is a similar formula for multivariate distributed systems in the full four block case. Here we are going to develop a rank type formula for the eigenvalues of A^*A. This is the subject of the next section.

4 Main Results

Let us begin with the following notation, assume that $W(z) = D(z)/k(z)$ and $G(z) = E(z)/k(z)$, where $D(z)$ and $E(z)$ are $N \times N$ matrices with polynomial entries of degree n, and $k(z)$ is a scalar polynomial of degree n. Then it is easy to see that ρ^2 is an eigenvalue of A^*A if and only if there exists a nonzero $x \in H^2(\mathbf{C}^N)$ such that

$$[\rho^2 k(S)^*k(S)I - E(S)^*E(S) - D(S)^*PD(S)]x = 0. \tag{1}$$

Note that $PD(S)x = D(\zeta)x - M(\zeta)P_{H^2}M(\zeta)^*D(\zeta)x$. Following the techniques used in the skew Toeplitz paper [1] we make the factorization:
$$M(\zeta)^*D(\zeta) = \Omega_d(\zeta)M_d(\zeta)^*, \tag{2}$$
for some $\Omega_d(\zeta)$ polynomial and $M_d(\zeta)$ inner, $N \times N$ matrices. In fact a procedure for the construction of this factorization is given in [2], and it is known that $det\ \Omega_d(\zeta) = det\ D(\zeta)$ and $det\ M_d(\zeta) = det\ M(\zeta)$. We now decompose $H^2(\mathbf{C}^N)$ as $H(M_d) \oplus M_dH^2(\mathbf{C}^N)$ and write $x = q + M_dx_d$ where $q \in H(M_d)$ and $x_d \in H^2(\mathbf{C}^N)$. Then we have
$$\begin{aligned} P_{H^2}M(\zeta)^*D(\zeta)x &= P_{H^2}\Omega_d(\zeta)M_d(\zeta)^*(q + M_dx_d) \\ &= \Omega_dx_d + P_{H^2}\Omega_d(\zeta)M_d(\zeta)^*q \\ &= M_d(\zeta)^*D(\zeta)M_d(\zeta)x_d + P_{H^2}\Omega_d(\zeta)M_d(\zeta)^*q. \end{aligned}$$
An upper bound for the degree of the entries of $\Omega_d(\zeta)$ is nN, and we can write $\Omega_d(\zeta) = \Omega_{d0} + \Omega_{d1}\zeta^1 + \cdots + \Omega_{dnN}\zeta^{nN}$. Since $q \in H(M_d)$, $M_d(\zeta)^*q \in L^2(\mathbf{C}^N) \ominus H^2(\mathbf{C}^N)$; therefore

$$P_{H^2}\Omega_d(\zeta)M_d(\zeta)^*q = \sum_{j=1}^{nN}\sum_{i=j}^{nN}\Omega_{di}\zeta^{i-j}q_{-j} =: q^\Omega(\zeta).$$

for some $q_{-j} \in \mathbf{C}^N$ $j = 1, \ldots, nN$. This will give us the following equality
$$M(\zeta)P_{H^2}M(\zeta)^*D(\zeta)x = D(\zeta)M_d(\zeta)x_d + M(\zeta)q^\Omega.$$

Thus
$$\begin{aligned} D(S)^*PD(S)x &= D(S)^*(D(S)x - D(S)M_dx_d - Mq^\Omega) \\ &= D(S)^*D(S)q - M_d\Omega_d(S)^*q^\Omega. \end{aligned}$$

This leads us to the following relationship:

$$[\rho^2k(S)^*k(S)I - E(S)^*E(S)]x - D(S)^*D(S)q = -M_d\Omega_d(S)^*q^\Omega. \tag{3}$$

Multiplying both sides by ζ^n and using the fact that $x = q + M_dx_d$ we obtain:
$$C(\zeta)q + B(\zeta)M_dx_d = \zeta^{n-1}u_{n-1} + \cdots + u_0 + M_d(\zeta)\tilde{\Omega}_d(\zeta)q^\Omega, \tag{4}$$
for some $u_0, \ldots, u_{n-1} \in \mathbf{C}^N$, where
$$\begin{aligned} B(\zeta) &:= \rho^2\zeta^nk^*(1/\zeta)k(\zeta)I - \zeta^nE^*(1/\zeta)E(\zeta) \\ C(\zeta) &:= B(\zeta) - \zeta^nD^*(1/\zeta)D(\zeta) \\ \tilde{\Omega}_d(\zeta) &:= \zeta^n\Omega_d^*(1/\zeta). \end{aligned}$$

We now summarize the above discussion with the following:

Proposition 2. ρ^2 is an eigenvalue of A^*A if and only if there exists a nonzero $x = q + M_dx_d \in H^2(\mathbf{C}^N)$ and some $u_0, \ldots, u_{n-1} \in \mathbf{C}^N$ such that (4) holds.

Proof. Immediate from above discussion. □

We now sketch the rest of our procedure. We basically follow a similar technique to the one developed in [4a]. First, we can write (4) in a form more suitable to the the multivariable nature of the problem. This may be done using a factorization similar to (2). Then taking the orthogonal projections on the subspaces of $H^2(\mathbf{C}^N)$ associated with the new inner matrix function obtained from this factorization, we will end up with two equations analogous to the equations (24c,d) of [4a]. This will bring us to a point where we can develop a rank type computation algorithm for the eigenvalues of A^*A. We give the details of these computations in the complete version of this paper [6].

5 Conclusions

In this paper we have studied H^∞ optimization of multivariate distributed systems. We took a special case, namely the multivariate two block version, of the more general four block problem. Here we described a procedure for the computation of the eigenvalues of this operator: A^*A. In the complete version of this paper [6] the specific rank type formula will be described, and some applications to the four block problem will be given. We also intend to include some specific distributed multivariate design examples as well.

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